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**LETTER TO THE EDITOR**

# Description of few-body systems via analytical continuation in coupling constant

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**Abstract.** The method of analytical continuation using the Padé approximant technique in the total and partial coupling constants is described and illustrated with some examples. The method can be applied to calculate the characteristics of the loose near-threshold and few-body resonance states, in the case of the potential with a strong repulsive core etc.

Many effective methods for studying the discrete spectrum of few-body systems with strong interaction have been developed including the various versions of perturbation theory, the Hartree–Fock method, the K-harmonic method, etc. Such methods (they will be referred to as the direct methods), however, are all effective in some region, whereas in other practically important cases (regions) the direct methods give slowly converging (or even diverging) expansions when, for example, a strong repulsive core (or other singularities) is present in the two-body potential and when one calculates the near-threshold and resonance states even in the case of smooth interaction potentials.

In this letter we will present a method based on a very simple concept which makes it possible to use the results of calculations of the characteristics of the system in the region where a direct method is most effective to find the characteristics of the real system when the interaction potential includes singularities or when the state of interest is near the threshold (or is a few-body resonance). The method uses the numerical analytical continuation of the studied characteristics (level energies, electromagnetic form factors and matrix elements of various operators) in the total or partial coupling constant by means of the Padé approximant (PA) technique (Baker 1965). We assume that some 'proper' method exists to calculate directly the eigen-energy $E_\lambda$ and corresponding eigenfunction $\Psi_\lambda$ at small coupling constant $\lambda$ (in this case the unperturbed Hamiltonian $H_0$ should include the smooth part of the interaction potential, and the singular part of the interaction, e.g. repulsive core, etc, should be taken as a perturbation $\lambda V$). Then, this value may be extrapolated to $\lambda = 1$ (or other necessary values of $\lambda$) on the basis of the PA $P_N(\lambda)/Q_M(\lambda)$ of type II (Baker 1965) using several initial values of the characteristic of interest which are found at small $\lambda$ by a direct method.

Such direct extrapolation, however, proves to be slowly converging in almost all cases (i.e., high accuracy requires high orders of $N$ and $M$, see for example the data in columns 2 and 4 of table 2). The accurate result can be obtained (and this is of major importance in the proposed approach) only by taking into account the analytical properties of the values to be continued in the complex $\lambda$ plane, which (as it has been shown) just ensures an accurate and stable extrapolation.

In the two-body case, consideration of the analytical properties (behaviour of the binding energy $E_2(\lambda)$ near the two-body threshold) indicates that the wavevector $K_2(\lambda) = (E_2(\lambda))^{1/2}$ should be continued instead of the energy $E_2(\lambda)$ (this relates to the bound and virtual states and to the resonance). For many-body systems, already several nearest thresholds should be included, i.e. the Riemann surface should be made uniform. It is true that the singularities on the few-body thresholds have not been known well up to now and require a more sophisticated analysis. As is shown in Yndurain (1971), however, the PA converge even on the cuts and, therefore, a convergence although slower, should also occur if the few-body singularities are neglected. Our three- and four-body calculations (see below) show that quite accurate and stable results can already be obtained when only two-body thresholds are included.

We now give some illustrative examples of our method.

(i) *Continuation in the coupling constant of the repulsive core*

Examples of two-, three-, and four-nucleon systems will be used to illustrate the Padé extrapolation in the coupling constant of the repulsive core for the two-body NN potential $S_1$ (Afnan and Tang 1968) with a strong repulsive core of the order of 1 GeV height.

(*a*) *Two-nucleon system, triplet.* Table 1 presents the comparison of extrapolated results for the energy, root-mean square radius and the charge form factor $F(q^2)$ with the accurate results†. It can be seen that to obtain an accurate extrapolation it is sufficient to know only several points at $\lambda \leqslant 0{\cdot}1$, i.e. at a core height of an order smaller than the true core; in this case the accuracy of the eventual result is only slightly worse than the accuracy of the initial data.

**Table 1.** Two-nucleon system, S′ potential.

| System | Two-nucleon, triplet | | | | Two-nucleon, singlet |
|---|---|---|---|---|---|
| Function to be approximated / Order of PA† | $K_2^t = \sqrt{E_2^t}$ ((MeV)$^{1/2}$) | RMS radius (fm) | $F(q^2)$ $q^2 = 1$ fm$^{-2}$ | $F(q^2)$ $q^2 = 2$ fm$^{-2}$ | $K_2^s = \sqrt{E_2^s}$ ((MeV)$^{1/2}$) |
| 1 | 1·39 | 3·74 | 0·215 | −0·012 | −1·18 |
| 2 | 1·49 | 3·74 | 0·267 | 0·062 | −0·171 |
| 3 | 1·49 | 3·75 | 0·267 | 0·062 | −0·370 |
| Value obtained in direct calculations | 1·49 | 3·76 | 0·278 | 0·065 | −0·367‡ |

† Data presented for the diagonal PA $[N, N]$ only.

‡ The value found from the effective-range theory has been presented as the accurate value (see the text).

(*b*) *Two-nucleon system, singlet.* Table 1 shows the comparison between the extrapolated result for the singlet deuteron pole and the result obtained using the effective-range theory for the $S_1$ potential. An algorithm of direct calculations of the

† Values obtained by directly solving the Schrödinger equation (using numerical or variational methods) are taken as the accurate results.

singlet pole position (avoiding the effective-range approximation) may prove to be fairly cumbersome.

(*c*) *Ground state of the three-nucleon system.* It is convenient here to transform to the function $K_3(\lambda)=(E_3(\lambda)-E_2(\lambda))^{1/2}$ for which the PA is constructed ($E_2(\lambda)$ and $E_3(\lambda)$ are the two-body and three-body binding energies respectively). The results are presented in table 2 (the results of the direct extrapolation for the function $E_3(\lambda)$ are presented for the sake of comparison). It can be seen that already the [1, 1] approximant ($\lambda_{max}=0{\cdot}1$) gives fairly accurate values of the three-body energy†.

**Table 2.** Three- and four-nucleon systems with S′ potential.

| System | Three-nucleon | | Four-nucleon | |
|---|---|---|---|---|
| Function to be approximated / Order of PA | $E_3$ (MeV) | $K_3=\sqrt{(E_3-E_2)}$ ((MeV)$^{1/2}$) | $E_4$ (MeV) | $K_4=\sqrt{(E_4-E_3)}$ ((MeV)$^{1/2}$) |
| 1 | 0·60 | 2·727 | 10·6 | 4·40 |
| 2 | 1·24 | 2·728 | 16·1 | 4·68 |
| 3 | 3·57 | 2·728 | 19·2 | 4·74 |
| 4 | 6·86 | 2·736 | 21·8 | 4·79 |
| Value obtained in direct calculations | 7·76 | 2·738 | 31·1 | 4·83 |

(*d*) *Ground state of* $^4He$. This system already contains two two-body thresholds (3+1 and 2+2). The results presented in table 2 show, however, that already the inclusion of the nearest threshold ($4\Rightarrow 3+1$) makes it possible to obtain a sufficiently accurate value of the energy. It can be seen from figure 1 that in this case also the calculations at a value of the core of an order smaller than the true core (broken curve) make it possible to correctly represent the entire function $K_4(\lambda)$ accurate to within several per cent at $\lambda = 1$.

The results obtained (for the three- and four-nucleon systems) may be improved by taking into account the next thresholds (both two and two-body ones).

(ii) *Resonances as analytical continuations of the bound states*

The method developed may be also applied, without substantial alterations, to the study of two- and few-body resonances. In this case the continuation may be made in both partial (in the case of a singular interaction, as above) and total coupling constants. Since, however, the resonance wavefunction is strongly peripherized, the sensitivity of the resonance width and energy to the strength of, for example, the short-range core will be very weak and, therefore, the continuation is more convenient in either the total coupling constant or the constant of the main part of the interaction.

The position of the two-body resonance pole in the *l*th partial wave is known to be determined by the complex zero $K_l(\lambda)$ of the Jost function $f_l(K)$ and near the

† The initial three- and four-body energies at small $\lambda$ were calculated using the stochastic variational method proposed recently by Kukulin and Krasnopol'sky (1975).

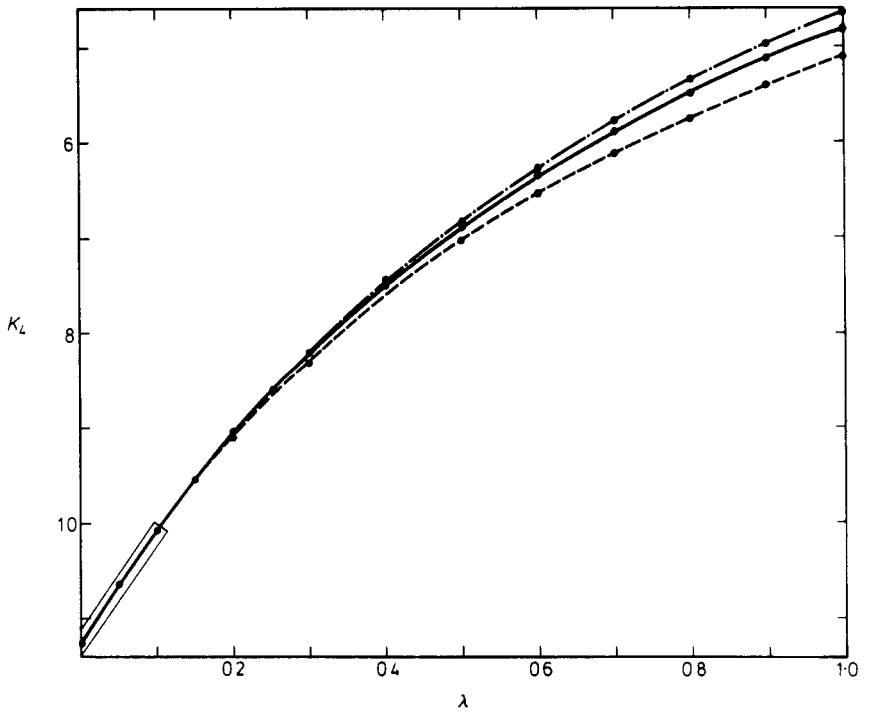


**Figure 1.** Energy of the ground state of the four-nucleon system. The full curve represents the exact value of the function $K_4(\lambda)=(E_4(\lambda)-E_3(\lambda))^{1/2}$. The broken curve is the PA $K_4^{[0,1]}(\lambda)=a/(1+b\lambda)$ (constructed using two points $K_4(\lambda=0)$ and $K_4(\lambda=0{\cdot}05)$). The chain curve is the PA $K_4^{[1,1]}(\lambda)=(a+b\lambda)/(1+c\lambda)$ (constructed using three points $K_4(\lambda=0)$, $K_4(\lambda=0{\cdot}05)$ and $K_4(\lambda=0{\cdot}1)$). The frame indicates the section of curve comprising the points on which both continuations have been constructed.

threshold one can derive the expansion (Demkov and Ostrovsky 1975)†:

$$K_l(\lambda)=\sum_{j=1}^{l}\mathscr{A}_j\lambda^{j-\frac{1}{2}}+\sum_{j=2l}\mathscr{C}_j\lambda^{j/2}. \tag{1}$$

The function $K_l(\lambda)$ given by the expansion (1) can be easily converted, using conventional techniques, into the PA. It should be noted only that near the threshold ($\lambda=0$) the inclusion of the leading term in (1) gives $K_l(\lambda)\sim\lambda^{1/2}$.

We have applied the above resonance calculation method to the $\alpha\alpha$ resonances $0^+$, $2^+$, $4^+$, and $6^+$, amongst others. The well known Ali–Bodmer $\alpha\alpha$ potential (Ali and Bodmer 1966) was used as the potential model. The resonance parameters (the resonance energy $E_R$ and width $\Gamma_R$) were found by calculating the *bound-state energies* in each partial wave at several values of the coupling constant $\lambda_i$; after that the PA were used to numerically continue the pole trajectory from bound states to the second energy sheet, and the actual resonance parameters were found at $\lambda=1$. In all the cases studied, quite accurate and numerically stable resonance parameters were found (see figure 2) which are in a good agreement with both experiment and values given by the $R$ matrix analysis of the corresponding theoretical $\alpha\alpha$ phase shifts. A similar approach can be

† A similar expansion has been obtained by Kukulin and Krasnopol'sky (1976) using other techniques.

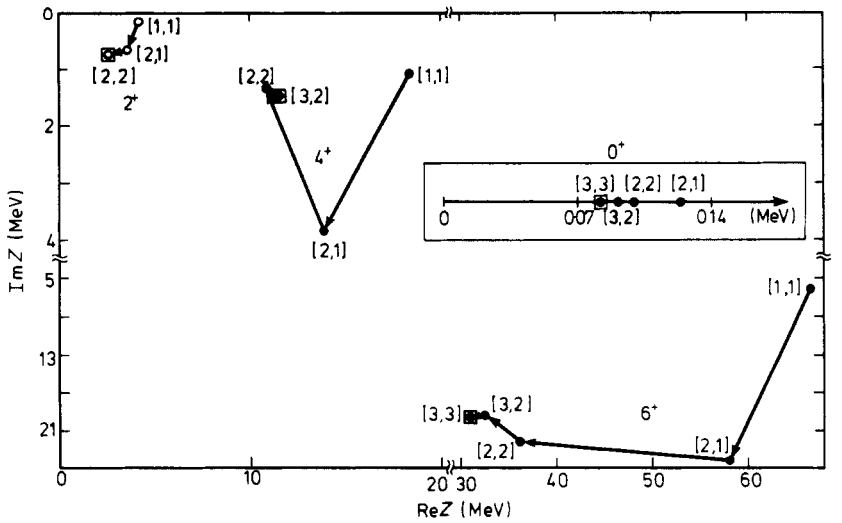


**Figure 2.** Positions of the $\alpha\alpha$ resonance poles with $J^{\pi} = 0^+, 2^+, 4^+$ and $6^+$ (the $0^+$ resonance is conveniently displayed in the top right-hand corner) on the second energy sheet at $\lambda = 1$ for the PA of different orders $[N, M]$ (the approximant order is indicated in square brackets). The square means that all the subsequent PA give a resonance pole located inside this square.

also applied to the analysis of few-body resonances and will be treated in detail in a subsequent publication.

It may be said in conclusion that the method we have suggested of analytical continuation in the coupling constant using PA of the second kind is a very convenient and effective means for studying few-body systems.

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